CHAPTER- 5

Fuzzy Logic

- *Learn the concept of fuzzy logic and its day-to-day utility*
- *Explain the functions of fuzzy arithmetic and operations*
- *Draw the structure of a fuzzy system*
- *List the applications of fuzzy logic systems*
- *Understand MatLab Implementation of Fuzzy Logic*
- *Recent trends in fuzzy technology*

5.1 Introduction: Fuzzy Logic

- *Introduced in 1965 by Lofti A. Zadeh, a professor in the University of California, Berkeley*
- *Got accepted as emerging technology since 1980s*
- *Having a fundamental trade-off between precision and cost which can be called "**principle of incompatibility**".*
- *Generalizes classical two valued logic for reasoning uncertainty.*
- *Helps us to tackle the uncertainty and vagueness associated with the event.*

5.2 Human Learning Ability, Imprecision and Uncertainty

- *The factors influencing humans learning ability are:*
  1. *Generalization*
  2. *Association (memory mapping)*
  3. *Information loss (memory loss or forgetfulness).*

5.3 Undecidability

- *Ambiguity that originates from the inability to distinguish between various states of an event is termed as* ***Undecidability***

5.4 Probability Theory versus Possibility Theory

- *Possibility measures the degree of ease for a variable to take a value, while probability measures the likelihood for a variable to take a value.*
- *So they deal with two different types of uncertainty.*
- ***Possibility theory handles imprecision and probability theory handles likelihood of occurrence.***
- *fuzzy set theory can define set membership as a possibility distribution.*
- *Also, fuzzy logic measures the degree to which an outcome belongs to an event while probability measures the likely hood of event to occur.*

5.5 Classical Sets and Fuzzy Set

- *'crisp set' is a collection of distinct (precisely defined) elements. In classical set theory, a crisp set can be a superset containing other crisp sets.*
- *Fuzziness is a property of language. Its main source is the imprecision involved in defining and using symbols.*
- *Fuzziness is a property of language. Its main source is the imprecision involved in defining and using symbols.*

5.5.1 Representation of a Classical Set

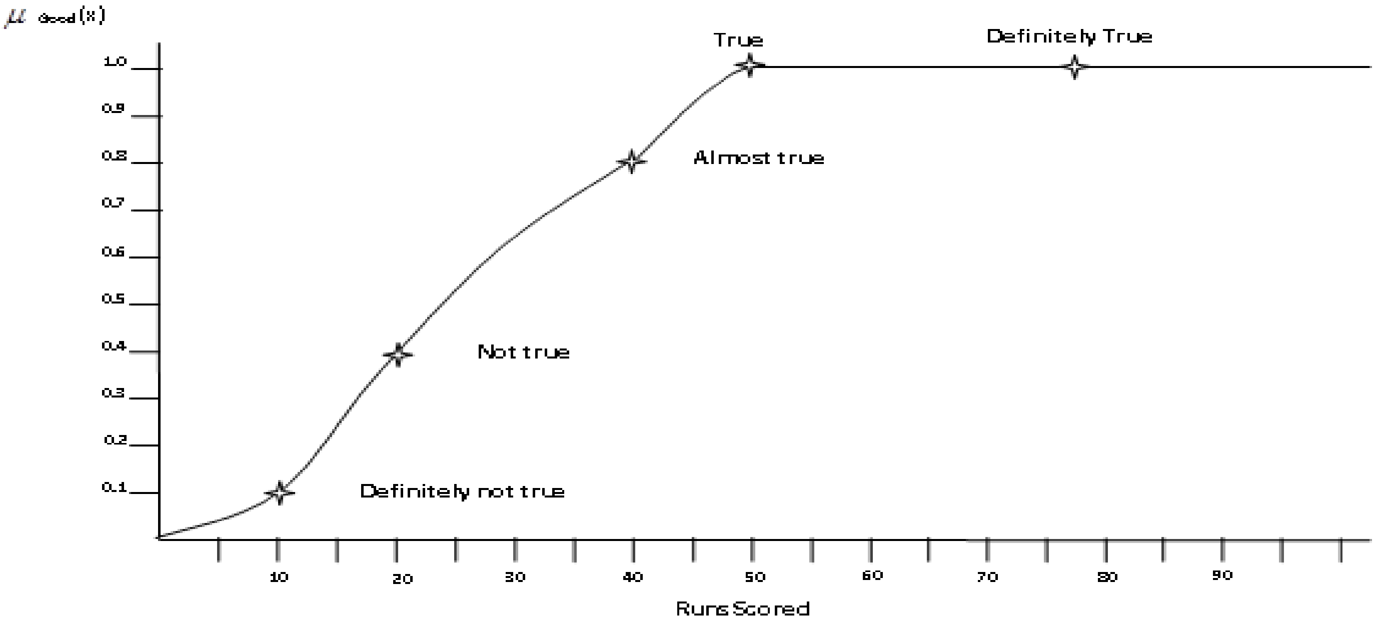
- *Classical set as already mentioned is a collection of objects of any kind. They can be represented as list method, rule method and characteristic function method.*

5.5.1 Representation of a Classical Set

Method	Examples
List method Listing of objects in a set parenthesis.	N={1,2,3,...100} A={Madras, Madurai, Roorkee, Mumbai}
Rule method The set is defined by property or rule satisfied by every objects within the set parenthesis.	N={x│f(x)} N= {f (3)} = {3, 6, 9.....} A={Densely populated city}
Characteristic function method The set is defined by a function termed as characteristic function denoted as μ . The characteristic function declares whether the element is a member of the set or not	A= {Pigeon, Horse, Trichy, 5, 6} x A implies that x is a member of set A x A implies that x is not a member of A. $\mu_A(x) = \begin{cases} 1 \text{ for } x \in A \\ 0 \text{ for } x \notin A \end{cases}$

5.5.2 Representation of a Fuzzy Set

- *Representation of a physical variable as a continuous curve is called* ***fuzzy set or membership function****.*
- *The numerical value is termed as* ***'Degree of membership' or 'Truth function'.***



5.5.2 Representation of a Fuzzy Set

- *Each pair (x, $\mu_A(x)$) is called a **singleton**. Here, x is followed by its membership in A, i.e. $\mu_A(x)$)*
- *Then the set A may be given by the collection or union of all singletons*
  A = {(1, 1.0), (2, 1.0), (3, 0.75), (4, 0.5), (5, 0.3), (6, 0.3), (7, 0.1), (8, 0.1)}
- *Alternative notation, the set of small integers mentioned above can be written as:*
  $A = \mu_A(1)/1 + \mu_A(2)/2 + \mu_A(3)/3 + \mu_A(4)/4 + \mu_A(5)/5 + \mu_A(6)/6 + \mu_A(7)/7 + \mu_A(8)/8$

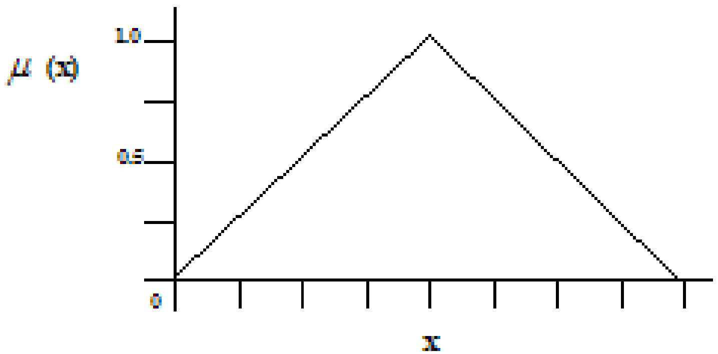
5.5.3 Basic Properties of Fuzzy Sets

- *The complete range of the model variable is called **'universe of discourse'**.*
- *Universe of discourse is associated with the system variable (score) and not with a particular fuzzy set (Good score, Average score or Bad score).*
- *The range of values covered by a particular fuzzy set is termed as **domain of fuzzy set.***
- *The domain of fuzzy set is set of elements whose degree of membership in the fuzzy set is greater than zero.*

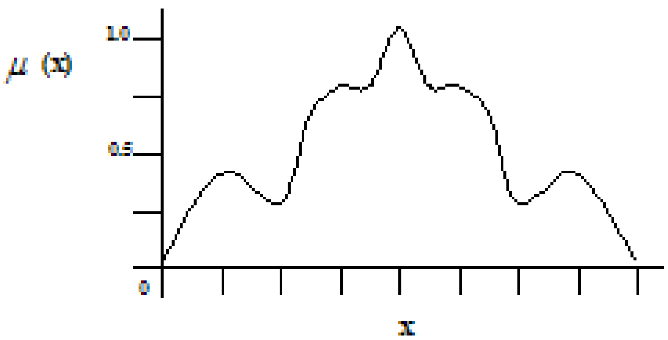
Non-convex and convex fuzzy sets

- *Non-convex fuzzy sets are fuzzy sets, in which membership grade alternately increases and decreases on the domain.*

- *Fuzzy sets in which membership grades do not alternately increases and decreases are called convex fuzzy set.*

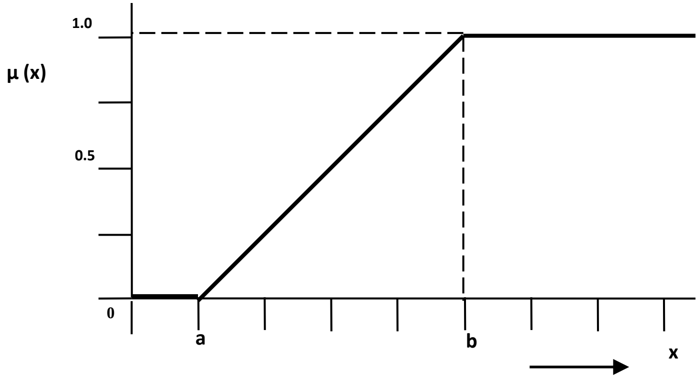


Convex fuzzy set



Non-convex fuzzy set

Some of the commonly used fuzzy sets are



$$\Gamma(x,a,b)=\begin{cases}0 & x<a\\ \dfrac{x-a}{b-a} & a\le x\le b\\ 1 & x\ge b\end{cases}$$

Fig. 5.7a Function fuzzy set (Linearly expressed gamma membership function)

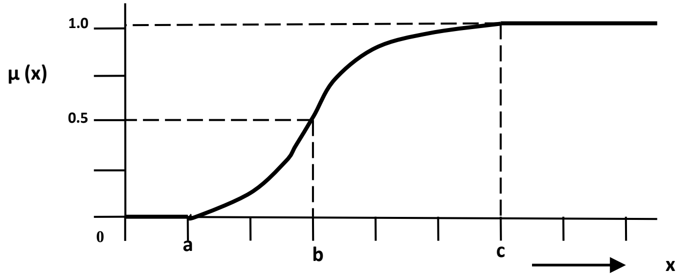


Fig. 5.7b Function fuzzy set (Sigmoidal membership function)

$$S(x,a,b,c)=\begin{cases}0 & x<a\\ 2\left(\dfrac{x-a}{c-a}\right)^2 & a\le x\le b\\ 1-2\left(\dfrac{x-c}{c-a}\right)^2 & b\le x\le c\\ 1 & x\ge c\end{cases}$$

Some of the commonly used fuzzy sets are

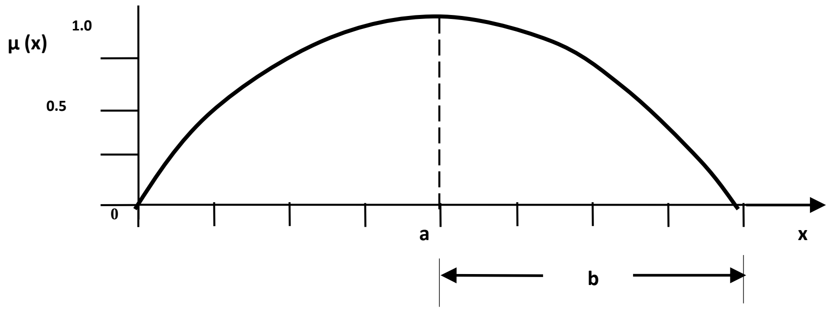


Fig. 5.7c Function fuzzy set (Gaussian membership function)

$$G(x,a,b) = e^{-b(a-x)^2}$$

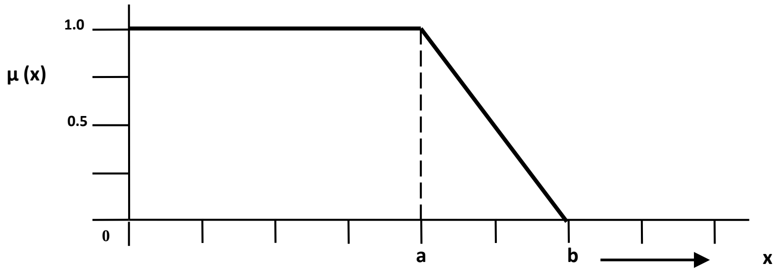


Fig. 5.7d Function fuzzy set (L membership function)

$$L(x,a,b) = \begin{cases} 1 & x < a \\ \dfrac{b-x}{b-a} & a \le x \le b \\ 0 & x \ge b \end{cases}$$

Some of the commonly used fuzzy sets are

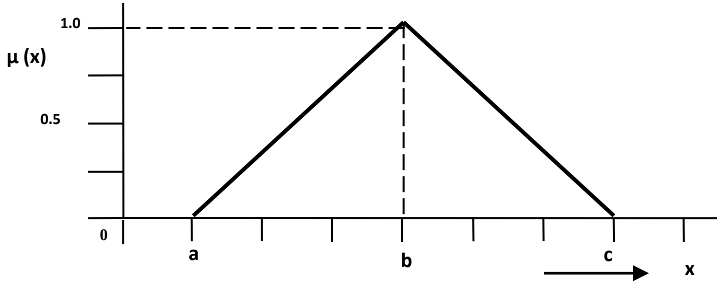


Fig. 5.7e Function fuzzy set (Triangle membership function)

$$\lambda(x,a,b,c) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \le x \le b \\ \dfrac{c-x}{c-b} & b \le x \le c \\ 0 & x \ge c \end{cases}$$

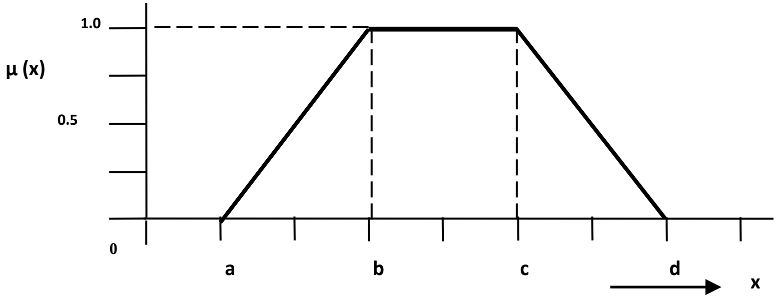


Fig. 5.7f Function fuzzy set (Trapezoidal membership function)

$$\Pi(x,a,b,c,d) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \le x \le b \\ 1 & b < x < c \\ \dfrac{d-x}{d-c} & c \le x \le d \\ 0 & x \ge d \end{cases}$$

5.6 Fuzzy Set Operations

- *Some of the important set operations that can be carried out using fuzzy sets are :*
  1. *Intersection of fuzzy sets*
  2. *Union of fuzzy sets*
  3. *Complement of fuzzy sets*

Intersection of Fuzzy Sets

- *Intersection of two fuzzy sets contains the elements that are common to both the sets and is also equivalent to logical AND operation.*
- *In fuzzy logic, this operation is performed by taking the minimum of the truth membership functions.*

$$\mu_A(x) \wedge \mu_B(x) = \min\{\mu_A(x), \mu_B(x)\}$$

5.6 Fuzzy Set Operations

Union of Fuzzy Sets

- *Union of two fuzzy sets contains all the elements belonging to both the sets and is equivalent to logical OR operation.*
- *In fuzzy logic, this operation is performed by taking the maximum of the truth membership functions.*

$$\mu_A(x) \vee \mu_B(x) = \max\{\mu_A(x), \mu_B(x)\}$$

Complement of Fuzzy Sets

- *The complement of a fuzzy set consists of the elements from universe which are not present in the set. This is equivalent to logical NOT operation.*
- *In fuzzy logic, this operation is performed by subtracting the membership value from 1.*

$$\mu_{\tilde{A}}(x) = 1 - \mu_A(x)$$

5.6.4 Important Terminologies in Fuzzy Set Operations

a) Empty Fuzzy Set

- *A fuzzy set A is said to be an empty set if it has no members and its membership function is zero everywhere in its universe of discourse U.*

$$A \equiv \emptyset \text{ if } \mu_A(x) = 0, \ \forall_x \in U$$

b) Normal Fuzzy Set

- *A fuzzy set A is said to be normal if it has a membership function that includes at least one singleton equal to unity in its universe of discourse U*

$$\mu_A(x_o) = 1$$

5.6.4 Important Terminologies in Fuzzy Set Operations

c) Equal Fuzzy Sets

- *If the membership functions of any two fuzzy sets A and B are equal everywhere in the universe of discourse, then the two fuzzy sets are said to be equal thereby satisfying the equation.* $A \equiv B$ if $\mu_A(x) = \mu_B(x)$

d) Fuzzy Set Support

- *A fuzzy set A is supported only if the crisp set of all $x \in U$ such that the membership values are non-zero.* $\mu_A(x) > 0$

d) Fuzzy Set Support

- *The product of the fuzzy set A and B produces a new fuzzy set, with its membership function value equal to the algebraic product of the membership function of A and B.*

$$\mu_{A \cdot B}(x) \equiv \mu_A(x) . \mu_B(x)$$

5.6.4 Important Terminologies in Fuzzy Set Operations

f) Fuzzy Set Multiplication by a Crisp Number

- *The membership function of a fuzzy set A is multiplied by the crisp number 'b' to obtain a new fuzzy set whose membership function $\mu_{bA}(x)$.*

$$\mu_{bA}(x) \equiv b \cdot \mu_A(x)$$

g) Power of a Fuzzy Set

- If α is set as the power of a fuzzy set A, then a new fuzzy set A^{α} has a membership function value

$$\mu_{A^{\alpha}}(x) \equiv [\mu_A(x)]^{\alpha}$$

5.6.5 Properties of Fuzzy Sets

Part A:

Double negation law	$\overline{(\overline{A})} = A$	
Idempotency	$A \cup A = A$	$A \cap A = A$
Commutativity	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Associative Property	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
Distributive Property	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Adsorption	$A \cap (A \cup B) = A$	$A \cup (A \cap B) = A$
De Morgan’s laws	$\overline{A \cap B} = \overline{A} \cup \overline{B}$	$\overline{A \cup B} = \overline{A} \cap \overline{B}$

5.6.5 Properties of Fuzzy Sets

Part B :

Law of Contradiction	$A \cap \overline{B} \neq \varnothing$	
Law of Excluded Middle	$A \cup \overline{A} \neq U$	
Intersection of a Fuzzy Set with an Empty Set	$A \cap \varnothing = \varnothing$	$\mu_A(x) \wedge 0 = 0$
Union of a Fuzzy Set with an Empty Set	$A \cup \varnothing = A$	$\mu_A(x) \vee 0 = \mu_A(x)$
Intersection of a Fuzzy Set with Universe of Discourse	$A \cap U = A$	$\mu_A(x) \wedge 1 = \mu_A(x)$
Union of a Fuzzy Set with Universe of Discourse	$A \cup U = U$	$\mu_A(x) \vee 1 = 1$

Some of the properties that are not valid for fuzzy sets but valid for crisp sets are

5.6.6 Fuzzy Arithmetics

1. Extension principle (discrete fuzzy set)

➢ *The extension principle is a mathematical tool for extending crisp mathematical notions and operations to the milieu of fuzziness. The extension principle suggests that fuzzifying the parameters of a function yields fuzzy outputs.*

(a) Addition $$\mu_c(z) \equiv \bigvee_{z=x+y} \left[\mu_A(x) \wedge \mu_B(y)\right]$$

(b) Subtraction $$\mu_{A-B}(z) \equiv \bigvee_{z=x-y} \left[\mu_A(x) \wedge \mu_B(y)\right]$$

(c) Multiplication $$\mu_{A.B}(z) \equiv \bigvee_{z=x.y} \left[\mu_A(x) \wedge \mu_B(y)\right]$$

(d) Division $$\mu_{A \div B}(z) \equiv \bigvee_{z=x \div y} \left[\mu_A(x) \wedge \mu_B(y)\right]$$

5.6.6 Fuzzy Arithmetics

2. Alpha Cut Method (Discrete Fuzzy Set)

- *The alpha cut method for discrete fuzzy set is carried out by performing arithmetic operations within the boundary elements present between fuzzy sets and the solutions that depend on the boundary values of the discrete fuzzy function*

(a) Addition : $A + B = [a_1^{(\alpha)}, a_2^{(\alpha)}] + [b_1^{(\alpha)}, b_2^{(\alpha)}]$, $A + B = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}]$

(b) Subtraction : $A - B = [a_1^{(\alpha)}, a_2^{(\alpha)}] - [b_1^{(\alpha)}, b_2^{(\alpha)}]$, $A - B = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}]$

(c) Multiplication: $A.B \equiv [a_1^{(\alpha)}, a_2^{(\alpha)}] . [b_1^{(\alpha)}, b_1^{(\alpha)}]$, $A.B = [a_1^{(\alpha)} . b_1^{(\alpha)}, a_2^{(\alpha)} . b_2^{(\alpha)}]$

(d) Division: $A \div B = [a_1^{(\alpha)}, a_2^{(\alpha)}] \div [b_1^{(\alpha)}, b_2^{(\alpha)}]$ $\quad A \div B = \left[\frac{a_1^{(\alpha)}}{b_2^{(\alpha)}}, \frac{a_2^{(\alpha)}}{b_1^{(\alpha)}} \right]$

5.6.6 Fuzzy Arithmetics

3. Alpha Cut Method (Continuous Fuzzy Set)

- *The alpha cut method for discrete fuzzy set is carried out by performing arithmetic operations within the boundary elements present between fuzzy sets and the solutions that depend on the boundary values of the discrete fuzzy function*

(a) Addition : $C = A + B = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}]$

(b) Subtraction : $C = A - B = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}]$

(c) Multiplication: $C = A_1 . B_1 = [a_1^{(\alpha)} . b_1^{(\alpha)}, a_2^{(\alpha)} . b_2^{(\alpha)}]$

(d) Division: $C = A \div B = \left[\frac{a_1^{(\alpha)}}{b_2^{(\alpha)}}, \frac{a_2^{(\alpha)}}{b_1^{(\alpha)}}\right]$

5.6.6 Fuzzy Arithmetics

4. Higher Order Arithmetic Operations

- *If the membership functions are non-linear or for higher order arithmetic operations, it is not easy to carry out arithmetic operations.*

- *In the first step, it has been assumed that the membership functions of the variables are at least linear and if not, then it has to be expressed in terms of approximate linear functions.*

- *In case of multiplication and division mathematical complexity i.e., the order of α in $a_1^{(\alpha)}$ is proportional to the order of multiplication and division. Hence, generation of suitable membership function became difficult.*

5.7 Fuzzy Relations

➢ *A fuzzy relation generalizes the aim of binary classical relation into one that allows partial membership. It is developed by allowing relationship between elements of two or more sets to take an infinite number of degrees of relation between extreme of "Completely Related" and "Non-Related ". It converts relation into matter of degree.*

5.7.1 Operations on Fuzzy Relations

(a) Union $$\mu_{R \cup S}(x, y) = \max(\mu_R(x, y), \mu_S(x, y))$$

(b)Intersection $$\mu_{R \cap S}(x, y) = \min(\mu_R(x, y), \mu_S(x, y))$$

(c) Complement $$\mu_{\overline{R}}(x, y) = 1 - \mu_R(x, y)$$

(d) Containment $$R \subset S \Rightarrow \mu_R(x, y) \leq \mu_S(x, y)$$

5.8 Fuzzy Composition

- *Fuzzy compositions can combine fuzzy relations defined on different Cartesian products with each other in a number of different ways. A more complex form of relation is called the 'composition of fuzzy relation', where the associations among fuzzy sets are of concern in addition to associations among the elements of each fuzzy set.*

Max-Min Composition:

$$R_1 \ o \ R_2 \equiv \int_{x \, x \, z} \underset{y}{V} \left[\mu_{R_1}(x, y) \wedge \mu_{R_2}(y, z)\right] / (x, z)$$

$$\mu_{R_i \ o \ R_2}(x, z) = \underset{y}{V} \left[\mu_{R_1}(x, y) \ \wedge \ \mu_{R_2}(y, z)\right]$$

Max-Star Composition:

$$R_1 * R_2 \equiv \int_{xxz} \underset{y}{V} \left[\mu_{R_1}(x, y) * \mu_{R_2}(y, z)\right] / (x, z)$$

$$\mu_{R1*R2}(x, z) = \underset{y}{V} \left[\mu_{R_1}(x, y) * \mu_{R_2}(y, z)\right]$$

Max-Product Composition:

$$R_1 . R_2 \ \equiv \int_{X \times Z} \underset{y}{V} \left[\mu_{R_1}(x, y). \mu_{R_2}(y, z)\right] / (x, y,)$$

$$\mu_{R_1 . R_2}(x, z) = \underset{y}{V} \left[\mu_{R_1}(x, y) . \mu_{R_2}(y, z)\right]$$

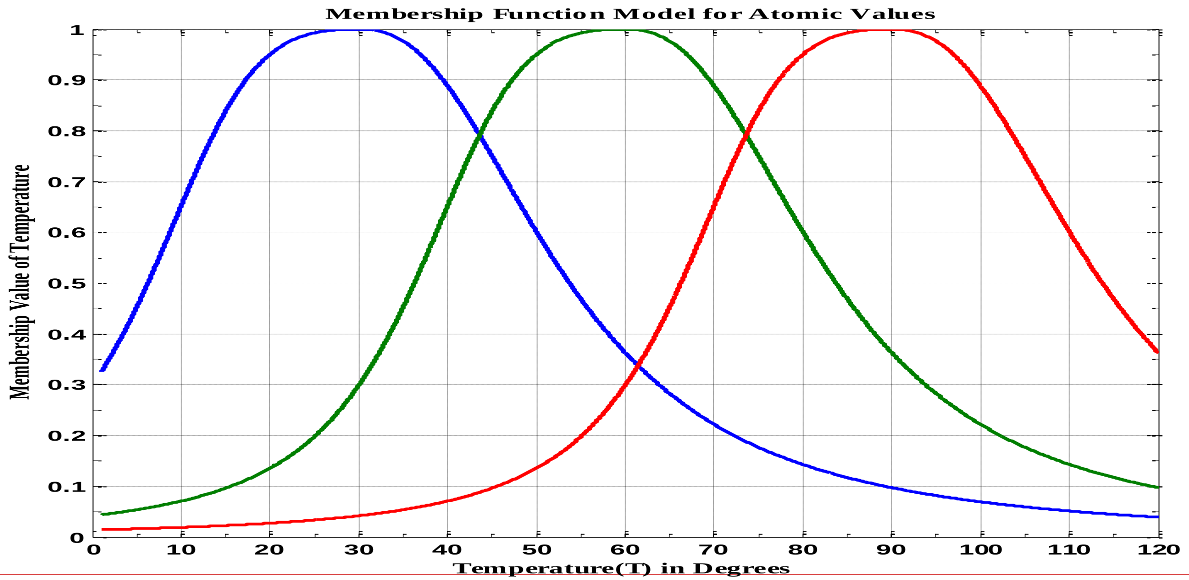
Max-Average Composition:

$$R_1 \langle + \rangle R_2 \equiv \int_{X \times Z} \underset{y}{V} \left[\frac{1}{2}\left(\ \mu_{R_1}(x, y) + \mu_{R_2}(y, z) \ \right)\right] / (x, z)$$

$$\mu_{R_1 \langle + \rangle R_2}(x, z) = \underset{y}{V} \left[\frac{1}{2}\left(\mu_{R_1}(x, y) + \mu_{R_2}(y, z)\right)\right]$$

5.9 Natural Language and Fuzzy Interpretations

- *A collection of these primary terms will form phrases, of our natural language. Examples of some atomic terms are slow, medium, young, beautiful, angry, cold, temperature etc.*
- *The collection of atomic terms can form compound terms. Examples of compound terms are very cold, medium speed, young lady, fairly beautiful picture, etc .*



5.9.1 Linguistic Modifiers

Primary Terms with Modifiers	Modification	Membership Function
Very	δ^2	$\int_I \frac{[\mu_\delta(x)]^2}{x}$
Very—_very	δ^4	$\int_I \frac{[\mu_\delta(x)]^4}{x}$
More or less Slightly	$\delta^{\frac{1}{2}}$	$\int_I \frac{[\mu_\delta(x)]^{\frac{1}{2}}}{x}$
Plus	$\delta^{1.25}$	$\int_I \frac{[\mu_\delta(x)]^{1.25}}{x}$
Minus	$\delta^{0.75}$	$\int_I \frac{[\mu_\delta(x)]^{0.75}}{x}$
Over	$1-\delta,\ x \geq x_{max}$ $0,\quad x < x_{max}$	$1 - \int_I \frac{\mu_\delta(x)}{x},\ x \geq x_{min}$ $0,\quad x > x_{min}$
Under	$1-\delta,\ x \leq x_{min}$ $0,\quad x > x_{min}$	$1 - \int_I \frac{\mu_\delta(x)}{x},\ x \leq x_{max}$ $0,\quad x < x_{max}$
Indeed	$2\delta^2,\quad 0 \leq \delta \leq 0.5$ $1-2[1-\delta]^2,\quad 0.5 \leq \delta \leq 1.0$	$2\int_I \left[\frac{\mu_\delta(x)}{x}\right]^2,\quad 0 \leq \int_I \left[\frac{\mu_\delta(x)}{x}\right] \leq 0.5$ $1-2\left[1-\int_I \left[\frac{\mu_\delta(x)}{x}\right]\right]^2,\ 0.5 \leq \int_I \left[\frac{\mu_\delta(x)}{x}\right] \leq 1.0$

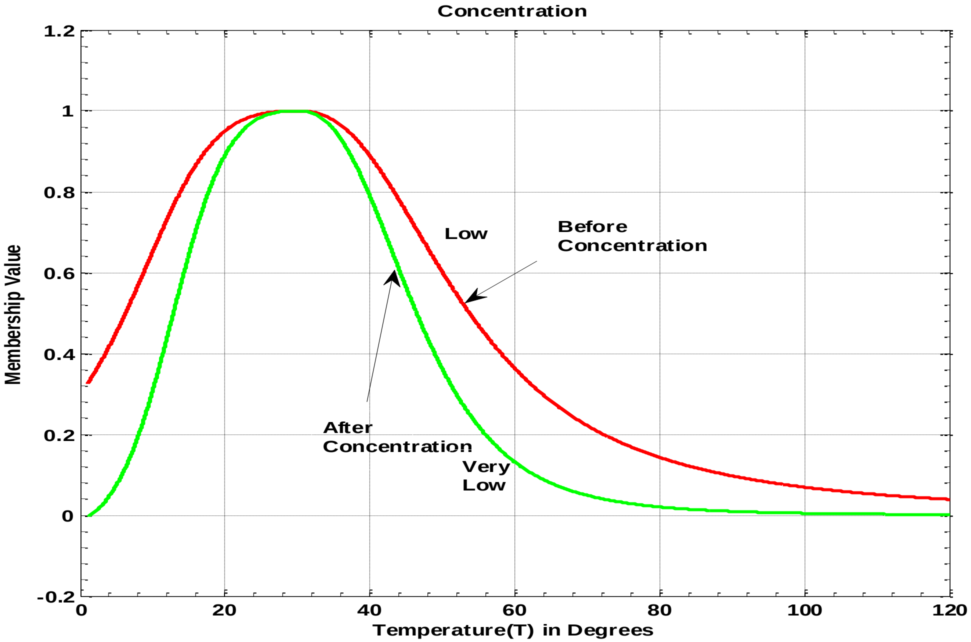
5.9.1 Linguistic Modifiers

1. Fuzzy Concentration

➢ *Concentrations tend to concentrate or reduces the fuzziness of the elements in a fuzzy set by reducing the degree of membership of all elements that are only "partly" in the set.*

$$\mu_A(\text{Low}, T) = \frac{1}{1 + 0.0005(T - 30)^{2.4}}$$

$$\mu_A(\text{Very Low}, T) = \left[\frac{1}{1 + 0.0005(T - 30)^{2.4}}\right]^2$$



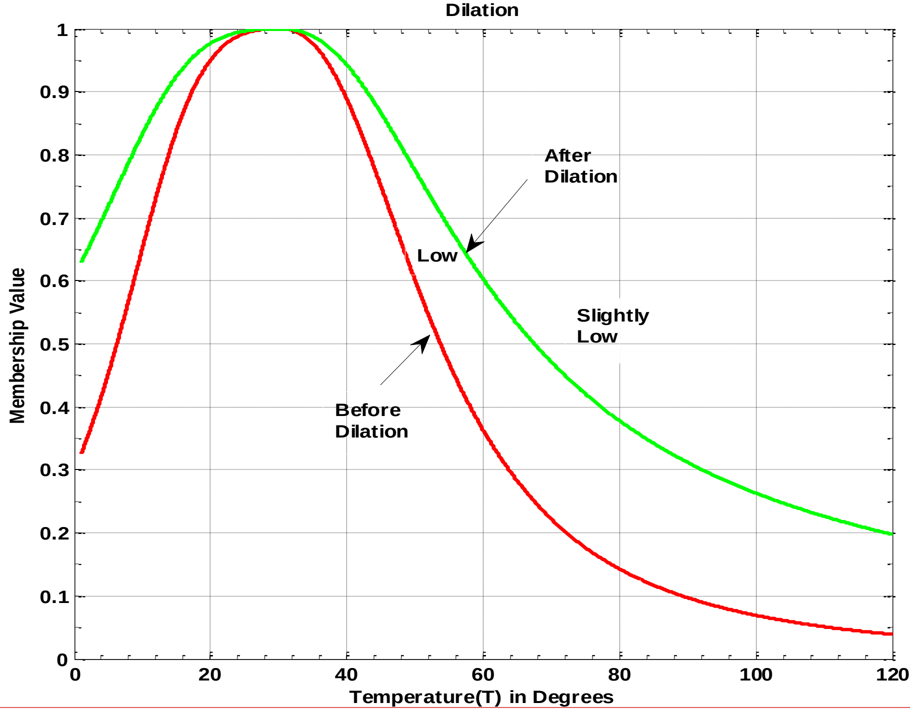
5.9.1 Linguistic Modifiers

2. Fuzzy Dilation

➢ *Dilation tend to dilate or increase the fuzziness of the elements in a fuzzy set by increasing the degree of membership of all elements that are only "partly" in the set.*

$$\mu_A(\text{Low}, T) = \frac{1}{1 + 0.0005(T - 30)^{2.4}}$$

$$\mu_A(\text{Slightly Low}, T) = \left[\frac{1}{1 + 0.0005(T - 30)^{2.4}}\right]^{0.5}$$



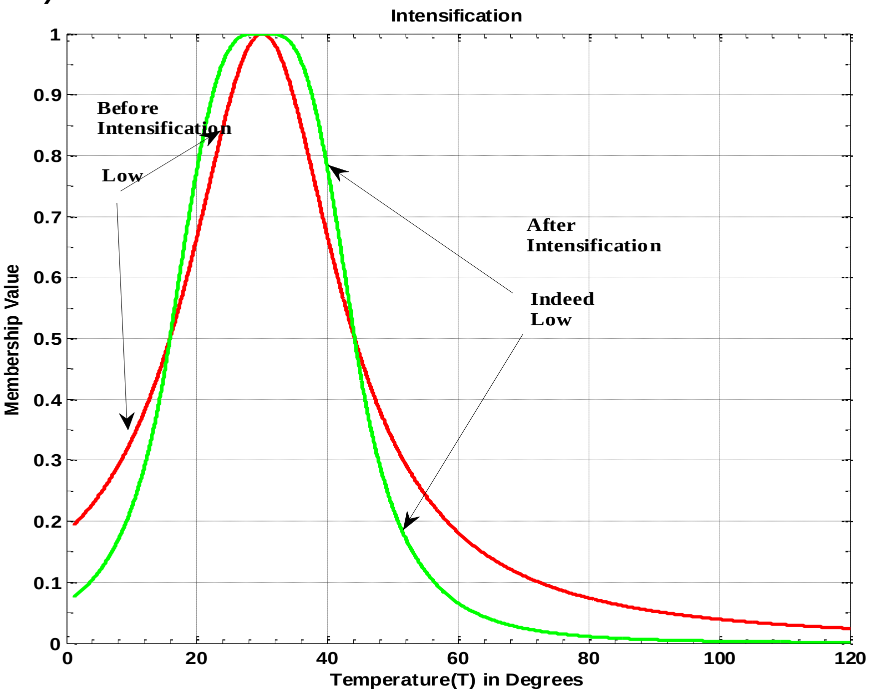
5.9.1 Linguistic Modifiers

3. Fuzzy Intensification

- *This property of the linguistic modifier is found to have a property which is a combination of both fuzzy concentration and fuzzy intensification.*

$$\mu_A(\text{Low}, T) = \frac{1}{1 + 0.0005(T-30)^{2.4}}$$

$$\mu_A(\text{Indeed Low}, T) = \begin{cases} 2 \times \left[\frac{1}{1+0.005(T-30)^{2.0}}\right]^2 & , 0 \le \left[\frac{1}{1+0.005(T-30)^{2.0}}\right] \le 0.5 \\ 1 - 2 \times \left[\frac{1}{1+0.005(T-30)^{2.0}}\right]^2 & , 0.5 \le \left[\frac{1}{1+0.005(T-30)^{2.0}}\right] \le 1.0 \end{cases}$$



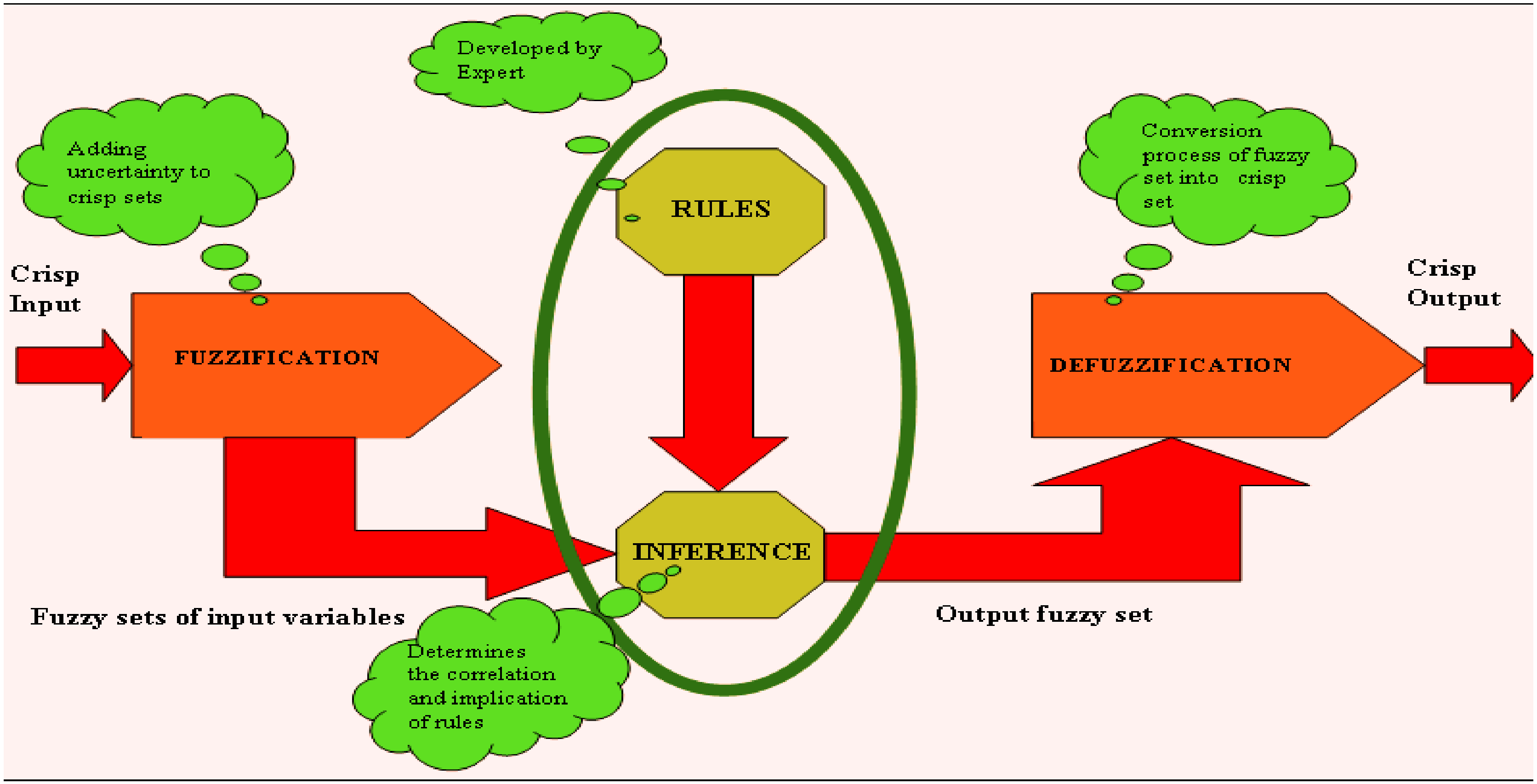
5.9.2 Logical Operations using Linguistic Modifiers

- *It is possible to perform logical operations using the linguistic modifiers, however the order of precedence need to be followed to avoid confusion. This problem can be resolved by properly using the parentheses by the rule that use "association to the right".*

- *The precedence of operation should be "NOT" in the First, "AND" in the second and "OR" in the third.*

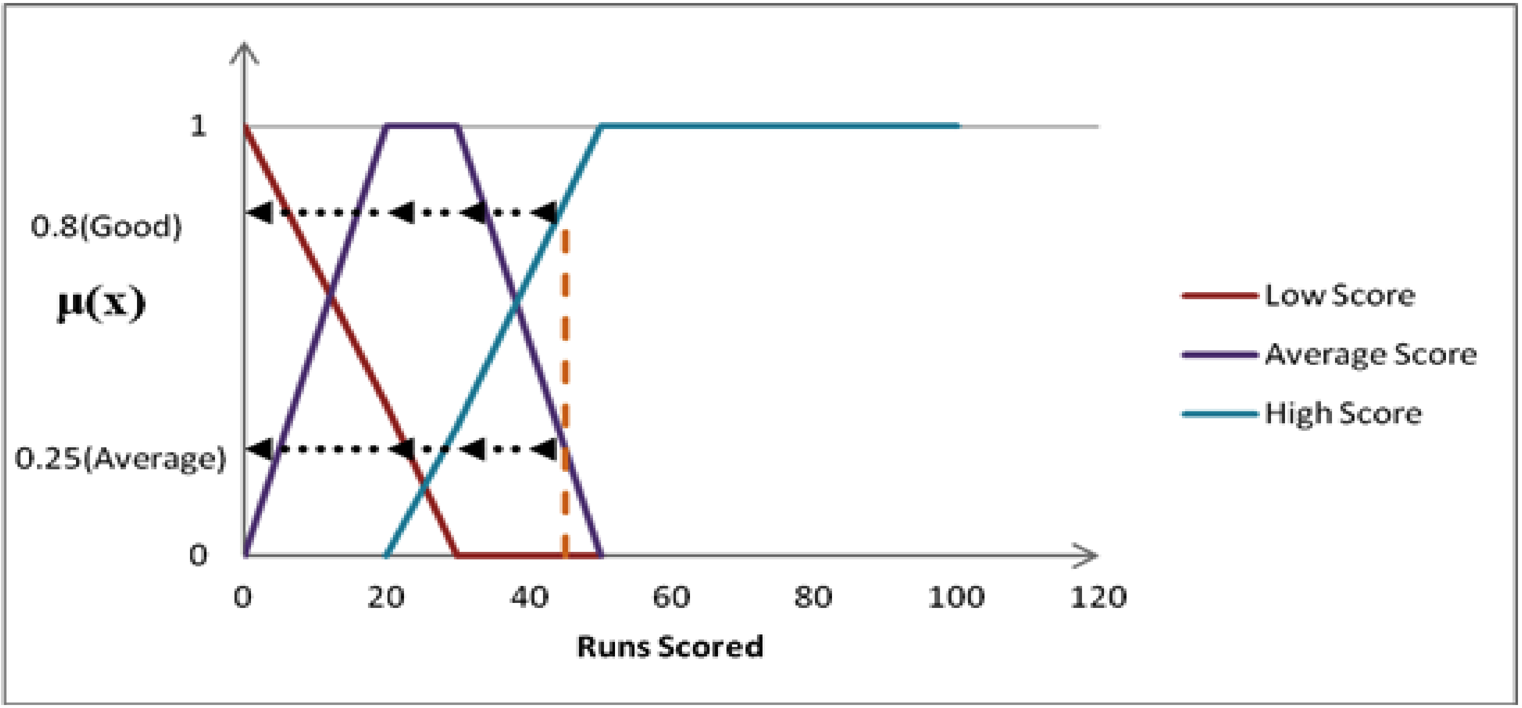
- *For example, if we have a compound terms "plus very minus very low", then following the rule "association to the right" while using the parentheses we get "Plus (very (minus (very (low))))".*

5.10 Structure of Fuzzy Inference System



5.10.1 Fuzzification

- *The process of converting the crisp values to fuzzy variable is called 'fuzzification'.*



5.10.2 Fuzzy Propositions

- *A sentence having only "true (1)" and "false (0)" as its truth value is called propositions which belongs to classical proportional logic.*

- *Fuzzy propositions can be classified into 4 types. They are:*

  (a) Unconditional and Unqualified Propositions:

  The canonical form of fuzzy propositions of this type, P, is expressed by the sentence

  $P : \mathcal{V}$ is F, E.g., Temperature of 45º is high

  (b) Unconditional and Qualified Propositions:

  Propositions P of this type are characterized by the canonical form

  $P : \mathcal{V}$ is F is S, *E.g., Temperature of 65º is high is very true*

  (c) Conditional and Unqualified Propositions:

  Propositions P of this type are expressed by the canonical form

  P : If $\mathcal{X}$ is A, then $\mathcal{Y}$ is B, *E.g. If the temperature is high, then it is hot*

  (d) Conditional and Qualified Propositions:

  Propositions P of this type can be characterized by the canonical form

  P : If $\mathcal{X}$ is A, then $\mathcal{Y}$ is B is S, *E.g. If the temperature is high, then it is hot is true.*

Fuzzy connectives

- *The linguistic variables are combined by various connectives such as negation, disjunction, conjunction and implication etc*

Symbol	Connective	Usage	Definition
-	Negation	$\overline{P}$	$1-T(P)$
$\vee$	Disjunction	$P \vee Q$	$\max(T(P), T(PP))$
$\wedge$	Conjunction	$P \wedge Q$	$\min(T(P), T(Q))$
$\Longrightarrow$	Implication	$P \Rightarrow Q$	$\overline{P} \vee Q = \max(1 - T(P), T(Q))$

5.10.3 Fuzzy Implication Relations

Implication Operator (Ψ)	Usage $\mu_R(x,y) = \psi(\mu_A(x), \mu_B(x)) =$	Interpretation of "else"
Zadeh Max-Min	$(\mu_A(x) \wedge \mu_B(y)) \vee (1-\mu_A(x))$	$\mathrm{AND}(\wedge)$
Mamdani Min	$\mu_A(x) \wedge \mu_B(y)$	$\mathrm{OR}(\vee)$
Larsen Product	$\mu_A(x) \bullet \mu_B(y)$	$\mathrm{OR}(\vee)$
Arithmetic	$1 \wedge (1-\mu_A(x) + \mu_B(y))$	$\mathrm{AND}(\wedge)$
Boolean	$(1-\mu_A(x)) \vee \mu_B(y))$	$\mathrm{AND}(\wedge)$
Bounded Product	$0 \vee (\mu_A(x) + \mu_B(y) - 1)$	$\mathrm{OR}(\vee)$
Drastic Product	$\begin{array}{lll} \mu_A(x), & \text{if} & \mu_B(y)=1 \\ \mu_B(x), & \text{if} & \mu_A(y)=1 \\ 0, & \text{if} & \mu_A(y)<1, \mu_B(y)<1 \end{array}$	$\mathrm{OR}(\vee)$
Standard Sequence	$\begin{array}{lll} 1, & \text{if} & \mu_A(x) \le \mu_B(y) \\ 0, & \text{if} & \mu_A(x) > \mu_B(y) \end{array}$	$\mathrm{AND}(\wedge)$
Gougen	$\begin{array}{lll} 1, & \text{if} & \mu_A(x) \le \mu_B(y) \\ \frac{\mu_B(y)}{\mu_A(x)}, & \text{if} & \mu_A(x) > \mu_B(y) \end{array}$	$\mathrm{AND}(\wedge)$
Godelian	$\begin{array}{lll} 1, & \text{if} & \mu_A(x) \le \mu_B(y) \\ \mu_B(y), & \text{if} & \mu_A(x) > \mu_B(y) \end{array}$	$\mathrm{AND}(\wedge)$

5.10.4 Fuzzy Inference Procedures

1. Generalized Modus Ponens (GMP)

- *GMP means "the way that affirms by affirming".*
- *GMP is generally stated as* IF x is A THEN y is B

$$\Rightarrow \frac{x \text{ is } A' \quad (\text{Analytically Known})}{y \text{ is } B' \quad (\text{Analytically Unknown})}$$

2. Generalized Modus Tollens (GMT)

- *GMT means "the way that denies by denying".*
- *GMT is generally stated as* IF x is A THEN y is B

$$\Rightarrow \frac{y \text{ is } B' \quad (\text{Analytically Known})}{x \text{ is } A' \quad (\text{Analytically Unknown})}$$

5.10.5 Fuzzy Inference Algorithms

- *Fuzzy inference algorithms are procedures that perform evaluation of fuzzy if/then rules*

  If x is A_1 then y is B_1 ELSE

  If x is A_2 then y is B_2 ELSE

  ……

  if x is A_n then y is B_n

Case I: IF/THEN rules with connective AND/OR

- *If x_1 is A_1 AND x_2 is A_2 AND…..AND x_n is A_n then y is B, can be represented by min or product operator*

  min(∧) preposition -$\mu(x_1,x_2,\ldots\ldots,x_n,y) = \Psi\,[\mu_{A1}(x_1) \wedge \mu_{A2}(x_2) \wedge\ldots\ldots \wedge \mu_{An}(x_n), \mu_B(y)]$ or

  product (.) preposition - $\mu(x_1,x_2,\ldots\ldots,x_n,y) = \Psi[\mu_{A1}(x_1)\cdot \mu_{A2}(x_2) \ldots\ldots \mu_{An}(x_n), \mu_B(y)]$

5.10.5 Fuzzy Inference Algorithms

Case II: Nested IF/THEN Rules

- *If x_1 is A_1 then (if x_2 is A_2 then….. (if x_n is A_n then y is B)….) can be represented by repeated application of an implication operator*

$$\mu(x_1,x_2,\ldots\ldots,x_n,y) = \Psi[\mu_{A1}(x_1), \Psi[\mu_{A2}(x_2) \ldots\ldots \Psi[\mu_{An}(x_n), \mu_B(y)]]$$

Case III: IF/THEN rules with connective ELSE

- *If x_1 is A_{1p} AND x_2 is A_{2p} AND…..AND x_{1n} is A_{np} then y is B_q ELSE, can be analytically described by an algorithmic relation of the form*

$$R_{\Sigma}(x_{1i}, x_{2i},\ldots\ldots x_{ni},y_j) = \sum_{(x_1,x_2,\ldots\ldots,x_{ni},y_j)} \mu_{\Sigma}(x_{1i},x_{2i},\ldots\ldots,x_{ni},y_j)/(x_{1i},x_{2i},\ldots\ldots,x_{ni},y_j$$

5.10.6 Defuzzification

- *Defuzzification is the process of converting the fuzzy set into a crisp value.*

  (a) Max-membership (Height method)

  (b) Centre of Area (COA) or Center of Gravity or Centroid Method

  (c) Weighted Average Method

  (d) Mean of Maxima (Middle of maxima)

  (e) Centre of Sums

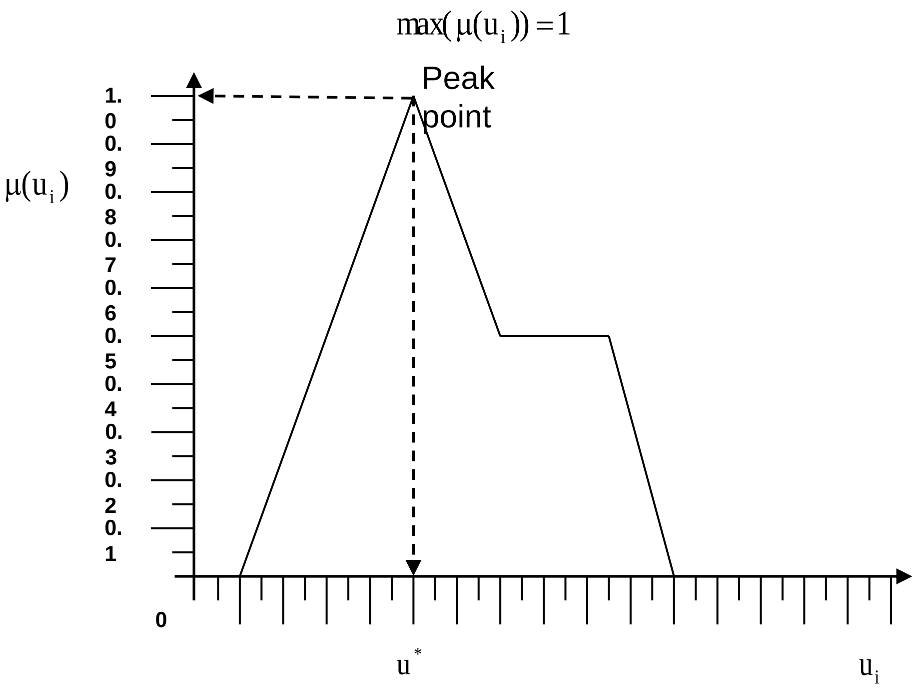
  (f) Centre of Largest Area (COLA)

  (g) First of Maxima

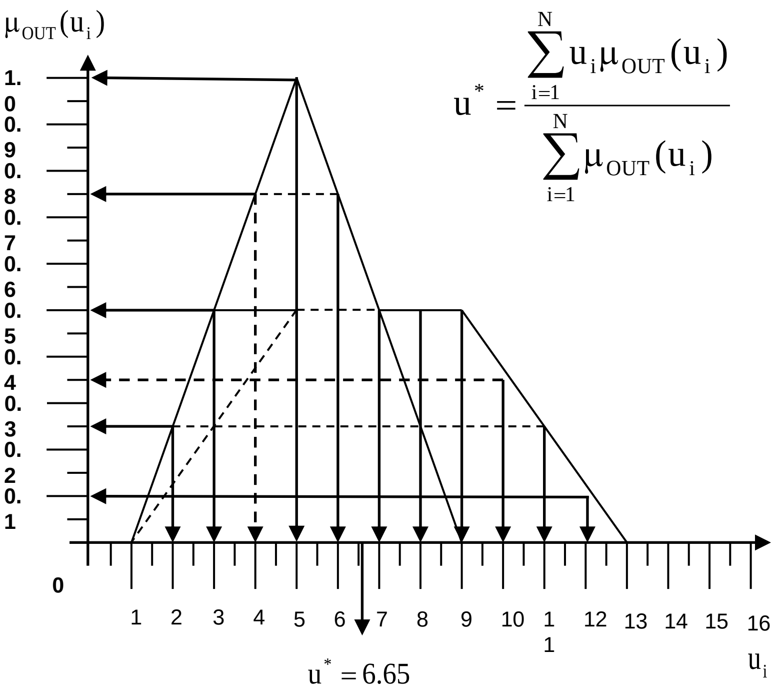
  (h) Last of Maxima

5.10.6 Defuzzification

a) Max-membership (Height method)



(b) Centre of Area (COA) Defuzzification

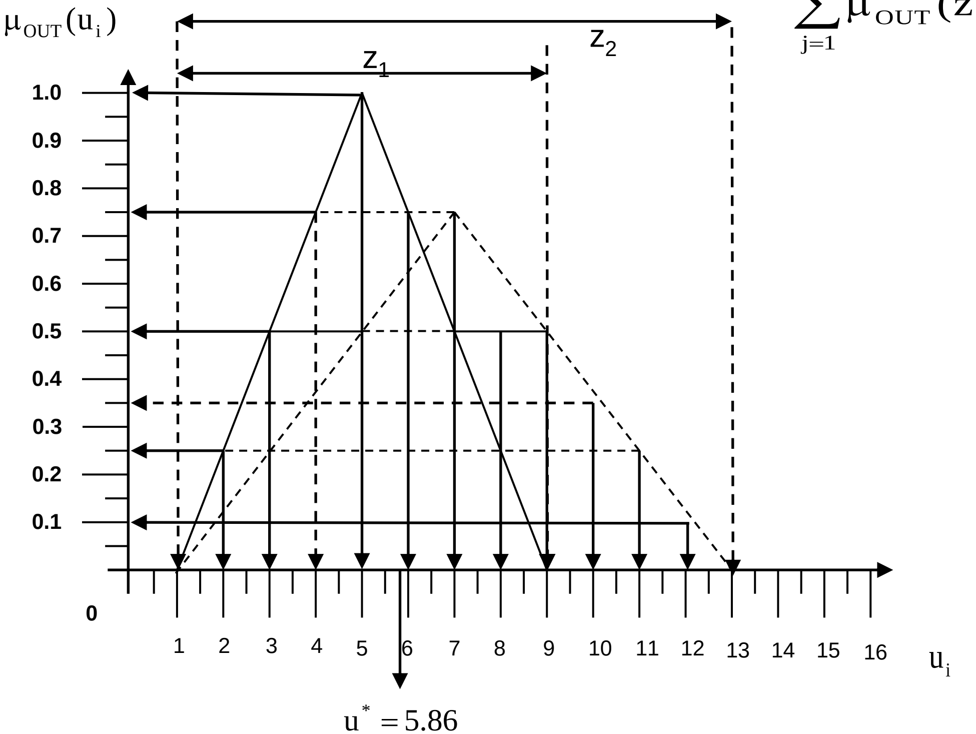


$$u^* = \frac{\sum_{i=1}^{N} u_i \mu_{OUT}(u_i)}{\sum_{i=1}^{N} \mu_{OUT}(u_i)}$$

5.10.6 Defuzzification

(c) Weighted Average Method

$$u^* = \frac{\sum_{j=1}^{Z} \bar{z}_j \mu_{OUT}(\bar{z}_j)}{\sum_{j=1}^{Z} \mu_{OUT}(\bar{z}_j)}$$



(d) Mean of Maxima (MOM)

$$u^* = \frac{1}{M} \sum_{m=1}^{M} u_m$$

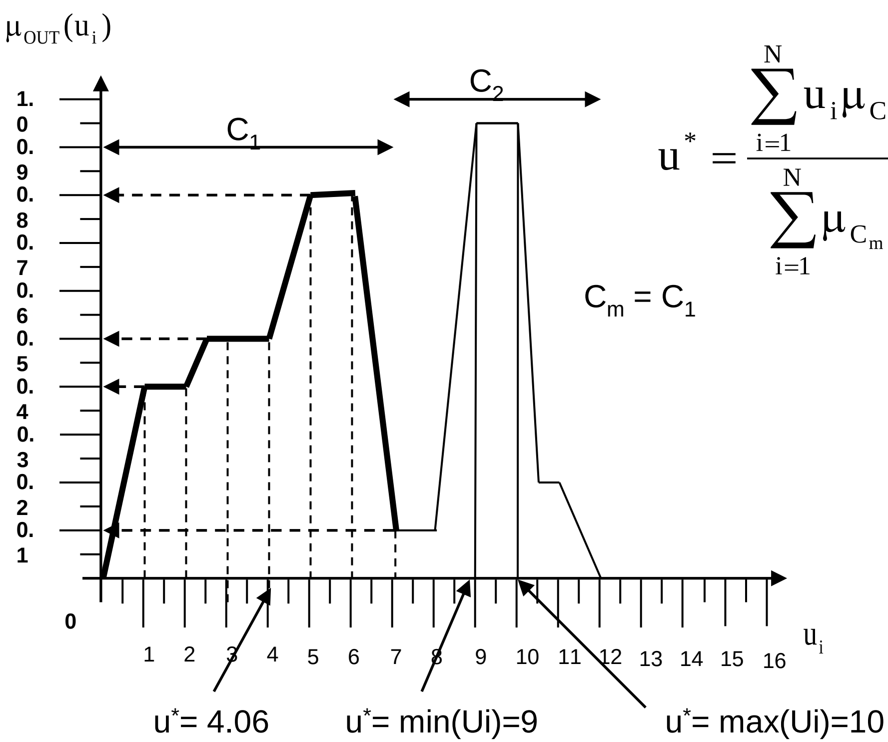
(e) Centre of Sums (COS)

$$u^* = \frac{\sum_{i=1}^{N} u_i \sum_{k=1}^{n} \mu_{C'_k}(u_i)}{\sum_{i=1}^{N} \sum_{k=1}^{n} \mu_{C'_k}(u_i)}$$

5.10.6 Defuzzification

(f) Centre of Largest Area (COLA)

$$u^* = \frac{\sum_{i=1}^{N} u_i \mu_{C_m}(u_i)}{\sum_{i=1}^{N} \mu_{C_m}(u_i)}$$



(g) First of Maxima

$$\max(\mu_{C_m}(u)) \geq \max(\mu_{C_i}(u)),\ i = 1....N_C$$

$$u^* = \min(u_i)$$

(h) Last of Maxima

$$u^* = \max(u_i)$$

Assessment of defuzzification methods

- *The ideal defuzzification technique should satisfy the following 5 criteria .*

    (a) Continuity

    (b) Disambiguity

    (c) Plausibility

    (d) Computational Complexity

    (e)Weighting Methods

- *It should be noted that none of the defuzzification techniques satisfy all the 5 criteria.*

- *However, the right choice of the defuzzification techniques for a particular application can be selected only by understanding the trade-off between these criteria.*